ON THE LINEARIZED THEORY OF FLOW AROUND BODIES. METHOD OF FORCE SOURCES

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PMM Vol.30, № 6, 1966, pp. 1006-1014

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(Received September 6, 1965)

The method of force sources is proposed for solving linear problems related to the interaction between rigid bodies, and fluids, or gases. Method is based on the introduction of perturbation force sources into equation of motion of fluid media. Boundary conditions at the rigid body surface make it possible to reduce the problem of hydrodynamic reactions to an integral equation defining the function of force sources. Method is illustrated by the solution of three simple problems in the field of acoustics, and of viscous, and compressible media flow around bodies.

In the linearized theory of flow around rigid bodies, as well as in acoustics, an important part of the sound wave generation analysis concerns the determination of hydrodynamic reactions of the medium on moving, pulsating, or oscillating bodies. Such reactions make themselves felt as constant, or variable mechanical forces, such as drag and lift, or in the case of sound wave emitters, as the wave resistance. Various methods had been proposed for the computation of such forces, as for example, in the monographs [1 to 6].

Here, a different approach to the problem of determination of surface forces exerted by liquids and gases on the rigid body is proposed. By resorting to the formalism of the generalized functions it is possible to introduce into the equations of motion of fluid media a perturbation source in the form volume density of forces exercised by the body on the gas. The distribution of surface tension entering into the expression of this force is selected in such a manner as to satisfy boundary conditions at the body surface. It becomes possible with the use of this device to reduce the problem of determination of forces acting on the body surface to the solution of certain integral equations. The proposed method is in all respects completely analogous to the well-known method of sources and sinks [1 to 4]. Both methods reduce the problem of interaction between body and gas to the solution of integral equations. The method of sources and sinks, however, leads to an integral equation which describes the distribution of fictitious sources and sinks in the volume of the body having the density of the medium, while the method of force sources yields an integral equation which directly defines the distribution of mechanical forces over the surface of the body (*).

We may note that the method of force sources had to a certain extent been

*) It is, of course, assumed here that the body does not have any special ejection devices, and that any strong surface evaporation of the body material, such as may occur at hypersonic velocities, is absent.

already used in papers [6 and 7] for the determination of sound radiation by means of point-force sources.

1. Basic equations and perturbation force sources. The system of linear equations defining small perturbations in a viscous compressible medium can be presented in the following form [8]:

$$\rho_0 \frac{d\mathbf{v}}{dt} = -\operatorname{grad} p + \eta \Delta \mathbf{v} + \left(\zeta + \frac{1}{3}\eta\right) \operatorname{grad} \operatorname{div} \mathbf{v} + \mathbf{f}(\mathbf{r}, \mathbf{t}) \qquad (1.1)$$

$$\frac{d\rho}{dt} + \rho_0 \operatorname{div} \mathbf{v} = Q(\mathbf{r}, \mathbf{t}), \qquad dp = c_s^2 d\rho \qquad \left(c_s = \left(\frac{\gamma p_0}{\rho_0}\right)^{1/2}\right) \qquad (1.2)$$

Boundary conditions at the rigid body surface S are

$$v_{nS} = V_0 \cos(\mathbf{n}, \mathbf{V}_0) \text{ for inviscid medium}, \quad \mathbf{v}_S = \mathbf{V}_0 \text{ for viscous medium} \qquad (1.3)$$

where ρ_0, p_0 and V_0 are respectively the density, pressure and velocity of the unperturbed medium; ρ, p and $\mathbf{v}$ are the respective perturbations of these parameters, η and ζ are viscosity coefficents, c is the velocity of sound in the medium; $d / dt = \partial/\partial t + (\mathbf{V}_0 \nabla)$ is the total derivative with respect to time in a coordinate system moving with velocity $\mathbf{V}_0$.

In the following we shall assume $Q = 0$, because of the stipulated absence of any special ejection devices and of body matter sutface evaporation.

We denote by $\mathbf{P}(\mathbf{r}, t)$ the force acting on a small surface area of the body. Using the generalized functions formalism [9 and 10], we may then write the expression of the body force per unit volume $\mathbf{f}$ as follows

$$\mathbf{f} = \mathbf{P}\delta[\varphi(\mathbf{r}, t)] \, |\operatorname{grad} \varphi| \qquad (1.4)$$

where $\varphi(\mathbf{r}, t)$ is the equation of the body surface which in the general case may pulsate, or oscillate as, for example, in problems of sound wave emission, $\delta[\varphi]$ is the Dirac function [9 and 10]. The resultant $\mathbf{F}$ of body forces acting on a gas is defined by the integral

$$\mathbf{F} = \iiint \mathbf{f} \, dx \, dy \, dz = \iint_S \mathbf{P} ds \qquad (1.5)$$

As is known from the general theory of linear differential equations, the solution of system (1.1), (1.2) can be presented in the form

$$v_i(\mathbf{r}, t) = \iiint_{-\infty}^{+\infty} \int G_{ik}(\mathbf{r}, t \,|\, \mathbf{r}', t') f_k(r', t') \, d\mathbf{r}' dt' \qquad (1.6)$$

where v_i and f_k are respectively the components of velocity and force; G_{ik} is the Green's tensor operator of the initial system (1.1),(1.2). Notation $(d\mathbf{r}' = dx'dy'dz')$ has been introduced for the sake of conveneience. Integration with respect to t' may be carried out either over the time interval from t_+ preceding the time of source action, or t_- following it.

Substituting (1.4) into Equation (1.6), and postulating the fulfilment of boundary conditions (1.3) over the body surface, we obtain the following integral equation for $\mathbf{P}(\mathbf{r}, t)$:

$$v_i(\mathbf{r}_S, t) = \iint\limits_S \int\limits_0^{t\pm} G_{ik}(\mathbf{r}_S, t \mid \mathbf{r}_S', t')\, P_k(\mathbf{r}_S', t')\, ds'dt' \tag{1.7}$$

The determination of the surface force $\mathbf{P}$ is thus reduced to the solution of integral Equations (1.7) with arguments in the form of Green's tensor function which is considered as known.

We note that in the derivation of (1.7) no assumptions were made as to the potential character of the interaction between body and medium, therefore, Equation (1.7) makes it possible to determine the full force acting on the medium which, in the general case, consistes of a turbulent and of a potential part. If, on the other hand, the sources and sinks $Q(\mathbf{r}, t)$ are left in the continuity Equation (1.2), and the stipulation is made that in (1.1) $\mathbf{f}(\mathbf{r}, t) = 0$, then, after appropriate computations, an integral equation may be derived from function $Q(\mathbf{r}, t)$. It is evident that this is, in fact, made on the assumption of a potential flow. It becomes, therefore, necessary when considering, for example, viscous media, to introduce into the analysis additional sources of turbulence. The metod of force sources considered here does not require such additional devices, making it possible to determine in a natural way both, the potential and the turbulent parts of the medium reaction forces. To illustrate the method of force sources we shall consider two simple problems.

The first problem deals with viscous flows. We shall find the perturbations which are created in a viscous incompressible medium by a circular cylinder of radius a and infinite length along the z-axis of the coordinate system (r, z, φ), rotating slowly about the axis with an angular velocity ω_0. We shall also determine the force resisting the cylinder rotation. The flow parameters are independent of φ, and Equation (1.1) becomes

$$\frac{d^2 v_\varphi}{dr^2} + \frac{1}{r}\frac{dv_\varphi}{dr} - \frac{v_\varphi}{r^2} = -\frac{f_\varphi}{\eta} \tag{1.8}$$

As the body surface equation is of the simple form $r = a$ we have from (1.4)

$$f_\varphi = \frac{T}{2\pi r}\,\delta(r - a) \tag{1.9}$$

Substituting the expression of f_φ into (1.5) and integrating with respect to φ and r, we come to the conclusion that the unknown constant T is the force acting on a unit of the cylinder length.

In this simple example it is possible to dispense with the preliminary determination of the Green's function. Applying the Hankel transformation to Equations (1.8) and (1.9), and using the inversion theorem [11], we obtain

$$v_\varphi(r) = \frac{T}{4\pi\eta} \times \begin{cases} a/r & \text{for } r \geqslant a \\ r/a & \text{for } r \leqslant a \end{cases} \tag{1.10}$$

For the determination of T we shall use boundary condition $v_\varphi(a) = a\omega_0$ which gives

$$T = 4\pi\eta a\omega_0 \tag{1.11}$$

This result is well known in hydrodynamics where it was derived by means of solving the boundary value problem of Equation (1.8) without its right-hand side [8].

As the second example, we shall consider a problem from the domain of acoustics concerning the reaction force of sonic waves emitted by a plate vibrating harmonically with frequency ω_0 and amplitude a in the direction of the z-axis perpendicular to the plate. For the displacement of particles of an inviscid, compressible medium we have from Equations (1.1) and (1.2) the following expression:

$$\frac{\partial^2 \xi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \xi}{\partial t^2} = \frac{1}{\rho_0 c^2} f_z(z, t) \tag{1.12}$$

The boundary condition at the plate surface is

$$\xi \mid_S = a \sin \omega_0 t \tag{1.13}$$

In accordance with (1.4) the expression of f_z is

$$f_z(z, t) = P(t)\, \delta(z - a \sin \omega_0 t) \tag{1.14}$$

where the unknown function $P(t)$ is the pressure on the plate surface. Green's function for Equation (1.12) in the case of a lagging argument is of the form

$$G(z, t \mid z', t') = 2\pi c H \left[t - t' - \frac{z - z'}{c} \right], \qquad H(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases} \tag{1.15}$$

Here, $H(x)$ is the Heaviside function [10].

Substituting relationships (1.13) to (1.15) into the general formula (1.7), integrating with respect to z', and differentiating the integral equation with respect to t, we obtain

$$\rho_0 a \omega_0 c \cos \omega_0 t = \int_{-\infty}^{+\infty} P(t')\, \delta \left[t - t' - \frac{a}{c} (\sin \omega_0 t - \sin \omega_0 t') \right] dt' \tag{1.16}$$

The solution of this integral equation is

$$P(t) = \rho_0 a \omega_0 c \cos \omega_0 t\, (1 - M \cos \omega_0 t) \qquad (M = a\omega_0 / c) \tag{1.17}$$

Here M is the Mach number. The appearance of the second harmonic in the expression of $P(t)$ is the result of the Doppler effect of the oscillating plate.

2. Volume force source of a simple form in a compressible flow. We shall consider the problem of perturbations generated in a fluid compressible medium by a volume-source force of the simplest form. We shall specify the simple force source of perturbations in a cylindrical coordinate system in the form of the following functions

$$\begin{aligned} \mathbf{f}(r, z, t) &= \frac{F}{2\pi a^2 l} \mathbf{e}_z \quad \text{for} \quad |z - V_0 t| \leqslant l, \ r \leqslant a \\ \mathbf{f}(r, z, t) &= 0 \qquad \text{in the remaining space} \end{aligned} \tag{2.1}$$

where the z-axis coincides with the direction of flow velocity V_0 and $\mathbf{e}_z$ is the unit vector in the same direction. The stipulation of form (2.1) for the force source means that the density of force $\mathbf{f}$ is throughout zero, with the exception only of the area bounded by a circular cylinder of radius a and length $2l$. Within the latter area the density of force $\mathbf{f}$ is directed along the z-axis and equal to the constant $F/2\pi a^2 l$. Using relationships (1.5), we conclude that the constant F is simply the drag of the perturbation force source, which will be approximately determined later.

The selection of form (2.1) for the force function was made not so much on physical considerations, as for the sake of comparative simplicity of further mathematical computations. Nevertheless, we may expect to derive certain information about the general character of an axial flow past bodies of rotation with dimensions a and $2l$, by using the results obtained for function $\mathbf{f}$ of form (2.1). It is highly probable that a distribution close to that of (2.1) would be realized with an axial flow past a three-dimensional axisymmetric cylindrical lattice body of radius a and length $2l$, constructed from slender bodies of rotation. The lattice pitch must be assumed to be considerably smaller than its dimensions, and the dimensions of the lattice forming elements, located at nodal points, to be smaller than the pitch. In this case force F will be the resultant of all forces acting on individual lattice elements. Expression of $\mathbf{f}$ in (2.1) may be considered as an asymptotic approximation to the actual distribution of the lattice forces per unit volume.

In a coordinate system in which the perturbation source (2.1) is assumed at rest, while the medium moves past it with velocity V_0 in the negative direction of the z-axis, we can derive from Equations (1.1),(1.2) the following relationship (*)

$$\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}+(1-M^2)\frac{\partial^2 p}{\partial \xi^2}=\frac{\partial f_z}{\partial \xi} \qquad (\xi=z-V_0 t,\ M=V_0/c) \quad (2.2)$$

In order to solve Equation (2.2) with function f_z defined by (2.1) we shall resort to a double Fourier-Hankel transformation [11]

$$\Psi_\nu(k,\varkappa)=\frac{1}{2\pi}\int_0^{+\infty}\int_{-\infty}^{+\infty} r\Psi(r,\xi)\,e^{i\varkappa\xi}J_\nu(kr)\,dr\,d\xi$$

with inversion formula

$$\Psi(r,\xi)=\int_0^{+\infty}\int_{-\infty}^{+\infty} k\overline{\Psi}_\nu(k,\varkappa)\,e^{-i\varkappa\xi}J_\nu(k,r)\,dk\,d\varkappa$$

where J_ν is a Bessel function, Ψ is any of the components of perturbation, or of function f_z, and $\overline{\Psi}_\nu$ the transform of function Ψ. Applying this integral transformation to Equation (2.2) and relationship (2.1), and using the inversion formula, we finally derive the following integral representation of pressure perturbations:

$$p=\frac{iF}{2\pi^2 al}\int_0^{+\infty}\int_{-\infty}^{+\infty}\frac{\sin(\varkappa l)\,J_0(kr)\,J_1(ka)\,e^{-i\varkappa\xi}}{k^2+(1-M^2)\varkappa^2}\,dk\,d\varkappa \quad (2.3)$$

The value of the double integral of (2.3) depends essentially on the Mach number. With the use of known methods [12] we obtain for subsonic velocities, when $M<1$

$$p(r,\xi)=\frac{F}{4\pi^2a^2l^2\gamma^2}\sum_{m=0}^{1}(-1)^m\Big\{\sqrt{(\xi\mp l)^2+\gamma^2(r+a)^2}\,E(q_m)-$$
$$-\frac{\gamma^2(r^2-a^2)}{\sqrt{(\xi\mp l)^2+\gamma^2(r+a)^2}}\Big[K(q_m)+\frac{(\xi\mp l)^2}{\gamma^2(r+a)^2}\Pi(q_m,-n)\Big]\Big\} \quad (2.4)$$

Here K, E and Π are complete elliptical integrals of the first, second and third kind respectively, with modulus q_m and parameter n defined by

$$q_m=2\gamma\left[\frac{ar}{(\xi\mp l)^2+\gamma^2(r+a)^2}\right]^{1/2},\quad n=\frac{4ar}{(r+a)^2} \qquad (\gamma=\sqrt{1-M^2}) \quad (2.5)$$

The minus sign of binomial $(\xi\mp l)$ applies to the first term of the sum in (2.4), and the plus sign to the second one. From (2.4) and (2.5) with $r=a$ we obtain at the surface of the cylinder bounding the source

$$p(a,\xi)=\frac{F}{4\pi^2a^2l\gamma^2}\sum_{m=0}^{1}(-1)^m\Big\{\sqrt{(\xi\mp l)^2+4a^2\gamma^2}\,E(q_m)-\frac{\pi}{2}|\xi\mp l|\Big\}$$
$$q_m=\frac{2a\gamma}{[(\xi\mp l)^2+4a^2\gamma^2]^{1/2}} \quad (2.6)$$

*) An analogous equation without right-hand side, derived for the velocity potential (the Prandtl-Glauert equation), was the subject of a detailed analysis in the monograph by Krasil'shchikova [2].

This pressure distribution is plotted on Fig.1 in dimensionless variables $y = 4\pi^2 a^2 \gamma^2 p/F$, $x = \xi/l$ for $\zeta = 2a\gamma/l = 0.1$.

It will be seen from this graph that for $M < 1$ there is an area of increased pressure upstream of the perturbation source, while at its rear we have an area of low pressure. In the lower left-hand corner of Fig.1 are shown experimental data obtained by Wood and Vincent for pressure distribution in an axial flow over bodies of rotation at subsonic velocities [13], with the distance on the abscissa axis. The similarity of these curves is easily seen.

With the use of asymptotic formulas for complete elliptic integrals, with $|\xi| \gg l$ and $r \gg a$, we obtain from (2.4) and (2.5) the following expression for the pressure away from the source

$$p(r, \xi) = \frac{F}{4\pi} \frac{\xi}{[\xi^2 + \gamma^2 r^2]^{3/2}} \tag{2.7}$$

It follows from this that the pressure distribution at some distance from the source is of dipolar character.

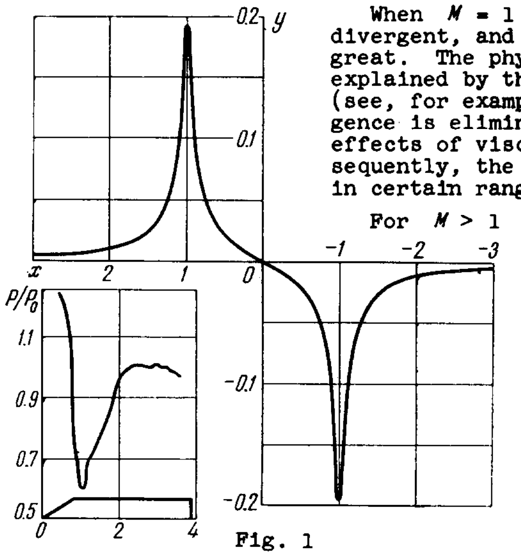

Fig. 1

When $M = 1$, the integral in Equation (2.3) is divergent, and the pressure tends to become infinitely great. The physical meaning of this can be simply explained by the beginning of sonic wave emission (see, for example, [16]). In real media this divergence is eliminated by nonlinear effects, and by the effects of viscosity and thermal conductivity. Consequently, the linearized theory becomes inadequate in certain ranges of nearsonic velocities.

For $M > 1$ the integral in (2.3) becomes again convergent, but the integrand has poles which are located on the real axis $\varkappa$ along which integration takes place. From the linear theory of radiation we know that in this particular case waves are generated by moving sources [6 and 9]. The presence of viscosity has the effect of shifting these poles of (2.3) into the upper half-plane of complex values of $\varkappa$. The improper integral of (2.3), as well as the functions by which these are expressed will be discontinuous [12]. The discontinuity lines are indicated on Fig.2, and bound areas 1 to 6:

$$l - \xi = \gamma_1 (r - a), \qquad l - \xi = \gamma_1 (r + a) \qquad (\gamma_1 = \sqrt{M^2 - 1}).$$
$$- l - \xi = \gamma_1 (r - a), \qquad l - \xi = \gamma_1 (r + a) \tag{2.8}$$

In the space $xy\xi$ these equalities define conical surfaces.

Computing the integral of (2.3) by the known method [12], we obtain in the case of parameter $\zeta = (\gamma_1 a/l) < 1$ the following expressions for pressure perturbations in the above area:

$$p_1 = 0 \quad p_2 = -\frac{F}{2\pi^2 a l \gamma_1}\left\{2\sqrt{\frac{r}{a}}\left[E(q_2) - \frac{\gamma_1^2 (r+a)^2 - (l-\xi)^2}{4ar\gamma_1^2} K(q_2)\right] + \frac{r^2 - a^2}{2a\sqrt{ar}}\left[K(q_2) - \frac{(l-\xi)^2}{\gamma_1^2 (r-a)^2} \Pi(q_2, n_2)\right]\right\}$$

$$p_3 = -\frac{F}{2\pi^2 a^2 l \gamma_1}\left\{\sqrt{(l-\xi)^2 - \gamma_1^2 (r-a)^2}\, E(q_3) - \frac{\gamma_1^2 (r^2 - a^2)}{\sqrt{(l-\xi)^2 - \gamma_1^2 (r-a)^2}}\left[K(q_3) - \frac{(l-\xi)^2}{\gamma_1^2 (r-a)^2} \Pi(q_3, n_3)\right]\right\} \tag{2.9}$$

$$p_4 = p_3(r, \xi) - p_2(r, -\xi), \qquad p_5 = p_3(r, \xi) - p_3(r, -\xi)$$

If $\zeta > 1$, we have

$$p_1' = 0, \qquad p_2' = p_2(r, \xi), \qquad p_3' = p_2(r, \xi) - p_2(r, -\xi)$$
$$p_4' = p_3(r, \xi) - p_2(r, -\xi), \qquad p_5' = p_3(r, \xi) - p_3(r, -\xi) \tag{2.10}$$

Here, the moduli and parameters of the complete elliptic integrals are defined by the following expressions:

$$q_2 = \left[\frac{(l-\xi)^2 - \gamma_1^2(r-a)^2}{4ar\gamma_1^2}\right]^{1/2}, \quad q_3 = \frac{1}{q_2}, \quad n_2 = \frac{(l-\xi)^2 - \gamma_1^2(r-a)^2}{\gamma_1^2(r-a)^2}$$
$$n_3 = \frac{4ar}{(r-a)^2} \tag{2.11}$$

Expressions for area 6, the trail area, where $r < a$ can be derived in a similar manner. Thus, all perturbations are concentrated within a circular cone with the apex angle $\varphi = 2\mathrm{tg}^{-1}(\gamma_1)^{-1}$.

Pressure distribution $p(\xi, a)$ on the surface of the cylinder containing sources with $\zeta < 1$ is defined by

$$p_1 = 0, \quad p_2 = \frac{F}{2\pi a l\gamma_1}\left\{q_2 + \frac{2}{\pi}\left[q_2 \cos^{-1} q_2 - E(q_2) + (1-q_2^2)K(q_2)\right]\right\}$$
$$p_3 = \frac{F}{2\pi a l\gamma_1}\left[q_2 - \frac{2}{\pi}E\left(\frac{1}{q_2}\right)\right], \qquad p_4 = p_3(\xi) - p_2(-\xi) \tag{2.12}$$
$$p_5 = p_3(\xi) - p_3(-\xi), \qquad q_2 = (l-\xi)/2a\gamma_1$$

Function $y = 2\pi a l\gamma_1 p/F$ has been plotted on **Fig.3** in terms of the dimensionless coordinate $x = \xi/l$ with parameter $\zeta = 0.1$. From Equation (2.9) we can, on the other hand, establish the asymptotic character of perturbations of pressure p at great distances from the sources, i.e. for $|\xi| \gg l$, and $r \gg a$. Thus, for example, we have in area 5

$$p_5(r, \xi) = -\frac{F}{2\pi}\frac{V_0 t - z}{[(V_0 t - z)^2 - \gamma_1^2 r^2]^{3/2}} \tag{2.13}$$

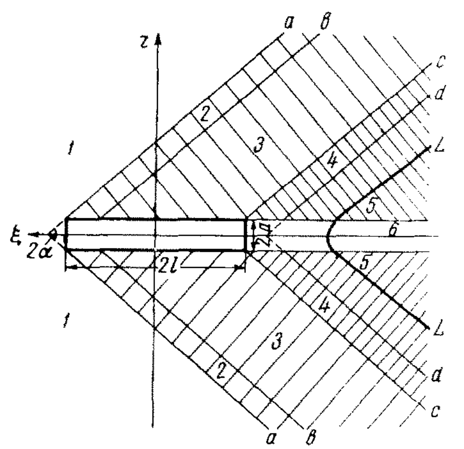

Fig. 2

A diagrammatic representation of $p_5(\theta)$ in spherical coordinates $r = R\sin\theta$, $\xi = R\cos\theta$ with constant R is given in Fig.2. It is interesting to note that the asymptotic formulas (2.7) and (2.13) coincide with expressions obtained earlier [6] for pressure perturbations created by a point-force source.

The magnitude of force F remained so far undefined. We shall now derive an approximate expression of this force in terms of the source characteristic dimensions a and $2l$, and of the Mach number. As the perturbation source is given in volume, and not at the surface, it is necessary for the determination of F to substitute for the surface boundary conditions a suitable condition as to the character of perturbations in the volume of the source action. It is evident that the medium is being slowed down in the area of the force source action. This is confirmed by experiments which show that downstream of the body rear face an area is formed where the gas is stationary, the area of stream stagnation [13 and 16]. In order to utilize this effect for the approximate determination of force F, we shall proceed from the equation for the z-components of velocity $w(\xi, r)$

$$-\rho_0 V_0 \frac{\partial w}{\partial \xi} = -\frac{\partial p}{\partial \xi} + f_z \tag{2.14}$$

We shall consider velocity perturbations at the flow axis, i.e. at $r = 0$. We integrate Equation (2.14) with respect to ξ within the limits of the area occupied by the source, and using (2.3) obtain

$$-\rho_0 V_0 [w(l, 0) - w(-l, 0)] = \frac{F}{\pi a^2} - \frac{2F}{\pi^2 al} \iint\limits_0^{+\infty} \frac{\sin^2(\varkappa l) J_1(ka)}{k^2 + (1 - M^2)\varkappa^2} dk\, d\varkappa \tag{2.15}$$

We assume that the velocity drop due to the slowing down of the flow is V_0 :

$$w(l, 0) - w(-l, 0) \approx -V_0 \tag{2.16}$$

With this we obtain from (2.15) the approximate expression of force F

$$F = \pi a^2 \rho_0 V_0^2 \left[1 - \frac{2a}{\pi l} \iint\limits_0^{+\infty} \left\langle \frac{\sin^2(\varkappa l) J_1(ka)}{[k^2 + (1 - M^2)\varkappa^2} \right\rangle dk\, d\varkappa\right]^{-1} \tag{2.17}$$

We introduce the coefficient of aerodynamic drag C , and tne relative thickness of the source α

$$C = \frac{F}{{}^1/_2\, \pi a^2 \rho_0 V_0^2}, \quad \alpha = \frac{a}{2l} \tag{2.18}$$

We compute the integral in Expression (2.17), and using notations of (2.18), obtain

$$C = \frac{2}{1 - (1 + \alpha\sqrt{1 - M^2} - \sqrt{1 + \alpha^2(1 - M^2)})/(1 - M^2)} \qquad \text{for } M < 1$$

$$C = \frac{2}{1 + (1 - \sqrt{1 - \alpha^2(M^2 - 1)})/(M^2 - 1)} \qquad \text{for } \alpha\sqrt{M^2 - 1} \leqslant 1 \tag{2.19}$$

$$C = \frac{2}{1 + 1/(M^2 - 1)} \qquad \text{for } \alpha\sqrt{M^2 - 1} \geqslant 1$$

Values of coefficient C computed for several values of M and three values of α are tabulated below

$M =$ 0.1	0.3	0.5	0.7	1.2	1.6	1.8	
$C =$ 2.103	2.108	2.120	2.147	2.000	2.000	2.000	($\alpha = 0.1$)
$C =$ 2.701	2.741	2.866	3.202	1.913	1.911	1.909	($\alpha = 0.3$)
$C =$ 3.248	3.364	3.689	4.749	1.772	1.754	1.740	($\alpha = 0.5$)

The drag coefficient C rapidly increases with increasing velocity, as long as $M < 1$, and slowly decreases with $M > 1$. This effect is called "the wave drag crisis". It is notable that the greater the value of parameter α , the earlier this effect becomes apparent. All this coincides qualitatively with experimental data on the dependence of coefficient C on M and α in axial flow past bodies of rotation [14].

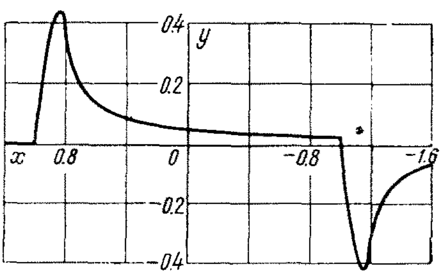

Fig. 3

The following interpretation of the dependence of the drag coefficient on the Mach number may be proposed. It is easily ascertained that the force source defined by (2.1) consists of turbulent and potential parts. It follows from Equation (1.1) that in the absence of viscosity effects, and with f_z specified by (2.1), the velocity vortex in the volume of the source and in its trail of radius a is not zero. It is clear from this that at slow subsonic velocities of the source, its drag is conditioned by the turbulence in its trail. In the subsonic velocity range of compressible fluids the effective drag due to turbulence appears to increase with increasing velocity in accordance with (2.19). With supersonic velocities the drag coefficient decreases, due to the emission of conical Mach waves which diminishes the turbulence drag.

We shall compute the emission intensity of sound waves generated by the force source when $M > 1$, by resorting to the method used in [6] for the case of a point source. Some simple calculations yield

$$\frac{dE}{dt} = \frac{F^2}{\pi\rho_0 a^2 l^2 V_0 (M^2-1)} \int_0^{+\infty} \frac{\sin^2 \varkappa l}{\varkappa^3} J_1^2(\alpha\varkappa\sqrt{M^2-1})\, d\varkappa \qquad (2.20)$$

(E is the emission energy)

The integrand of (2.20) is in essence the spectral density of sound emission intensity expanded into a Fourier series with respect to wave numbers. The frequency spectrum of emission is obtained from (2.20) by the simple transformation $\omega = \varkappa V_0$, where ω is the frequency of the emitted sonic wave. For $l \to 0$, and $a \to 0$ a divergent integral is obtained in the case of a point source [6]. The convergence of this integral in the case of an extended source is conditioned by the interference of emitted waves (see, for example, [17]). Unfortunately, it had not been possible to express the integral of (2.20) by any known function.

To evaluate the limit of applicability of the linear approximation we shall assume in our case that the area of the source action is fairly elongated, i.e. $\alpha < 1$. As a rough measure of applicability of the linear approximation we shall use the following condition:

$$p_{\max} = |p(-l, 0)| < p_0 \qquad (2.21)$$

where p_0 is the pressure in the undisturbed flow. Computing $p(-l, 0)$ with the aid of Formula (2.3) and substituting this into (2.21), we obtain

$$M < M_* = \sqrt{1-\alpha^2} \quad \text{for } M < 1, \qquad M^2\alpha^2 < 1 \quad \text{for } M > 1 \qquad (2.22)$$

These two criteria are known from the linearized theory of flow past bodies [13]. Thus the linear approximation can only reveal the tendency of the drag coefficient to grow with M approaching the critical value M_*. In the domain of $M_* < M \leq 1$ the linearized theory becomes inadequate, and it is necessary to take into account nonlinear effects, and the transport effects of viscosity and thermal conductivity.

It is hoped that the method of force sources proposed here may be found useful for the derivation of both rigorous and approximate solutions of problems concerning the interaction between rigid bodies and fluid media.

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Translated by J.J.D.